A General ANNV Family with a Common Special Kac-Moody-Virasoro Symmetry Algebra

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A general asymmetric Nizhnik-Novikov-Veselov (ANNV) family with an arbitrary function of high order group invariants is proposed. It is proved that the general ANNV family possesses a common infinite dimensional Kac-Moody-Virasoro symmetry algebra. The Kac-Moody-Virasoro group invariant solutions and the Kac-Moody group invariant solutions of the ANNV family are also studied. — PACS: 02.30.Jr, 02.30.Ik, 05.45.Yv.

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1. Introduction

How to construct the exact solutions of a given nonlinear differential equation plays an important role in soliton theory. There are many methods to obtain soliton solutions of nonlinear differential equations. One of them is the symmetry reduction method. From the symmetries of a nonlinear differential equation one can easily obtain many new solutions. In the study of (2+1)-dimensional integrable models it is found that for all the known integrable systems there is an isomorphic centerless Virasoro symmetry algebra ([1 – 4] and the references therein):

$$[\sigma(f_1),\sigma(f_2)] = \sigma(f_2\dot{f}_1 - f_1\dot{f}_2), \qquad (1.1)$$

where f_1 and f_2 are arbitrary functions of a single independent variable. The dot means the derivative of the functions with respect to their argument. In [5], a method was established to obtain the models with centerless Virasoro symmetry algebras. In this paper, we are concentrated on giving a possible asymmetric Nizhnik-Novikov-Veselov (ANNV) family

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + F(u) = 0, \qquad (1.2)$$

which possesses the same infinite dimensional Virasoro Lie point symmetry algebra (1.1), where

$$F(u) = F(u, u_x, u_{xx}, \cdots, u_{x^n y^m}, \cdots) \equiv F \qquad (1.3)$$

is an undetermined function of the field u and its any order derivatives of x and y, but not explicitly space-time dependent.

2. Review on the Lie Point Symmetries and Finite Transformations of the ANNV Equation

The Lie point symmetries and the related Kac-Moody-Virasoro algebra of the usual ANNV equation

$$p_t + p_{xxx} - 3(pr)_x = 0,\ r_y = p_x \qquad (2.1)$$

has been given by Tamizhmani and Punithavathi [6] by means of the standard classical Lie symmetry approach. In fact, by the transformations $r = u_x$ and $p = u_y$, we obtain the one variable ANNV equation

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} = 0. \qquad (2.2)$$

The equation (2.1) has been studied by Boiti et al. [7] and solved the Cauchy problem. The variable separable solutions are given in [8]. The Painlevé property for equation (2.2) has been proved by Dorizzi et al. [9].

A symmetry σ of the ANNV equation (2.2) is a solution of its linearized equation

$$\sigma_{yt} + \sigma_{xxxy} - 3\sigma_{xx}u_y - 3u_{xx}\sigma_y - 3\sigma_xu_{xy} - 3u_x\sigma_{xy} = 0, \qquad (2.3)$$

that means (2.2) is form invariant under the transformation

$$u \to u + \varepsilon\sigma,$$

where ε is an infinitesimal parameter.

According to the results of [6], the full Lie point symmetries of the ANNV equation (2.2) are the linear combinations of the following generators

$$\sigma_1(h) = h(t), \tag{2.4}$$

$$\sigma_2(g) = g(t)u_x + \frac{x}{3}\dot{g}(t), \tag{2.5}$$

$$\sigma_3(f) = x\dot{f}(t)u_x + 3f(t)u_t + u\dot{f}(t) + \frac{x^2}{6}\ddot{f}(t), \tag{2.6}$$

$$\sigma_4(l) = l(y)u_y, \tag{2.7}$$

where f, g, h are arbitrary functions of t and l is an arbitrary function of y. The nonzero commutation relations among (2.4), (2.5), (2.6) and (2.7) are given by

$$[\sigma_3(f_1), \sigma_3(f_2)] = 3\sigma_3(f_1\dot{f}_2 - \dot{f}_1 f_2), \tag{2.8}$$

$$[\sigma_3(f), \sigma_2(g)] = \sigma_2(3f\dot{g} - g\dot{f}), \tag{2.9}$$

$$[\sigma_3(f), \sigma_1(h)] = \sigma_1(3f\dot{h} + h\dot{f}), \tag{2.10}$$

$$[\sigma_2(g_1), \sigma_2(g_2)] = \frac{1}{3}\sigma_1(g_1\dot{g}_2 - g_2\dot{g}_1), \tag{2.11}$$

$$[\sigma_4(l_1), \sigma_4(l_2)] = \sigma_4(l_1\dot{l}_2 - l_2\dot{l}_1). \tag{2.12}$$

From (2.8) and (2.12) we know that the subalgebra constituted by σ_2 and/or σ_4 is just the Virasoro algebra (1.1).

The general finite transformation related to the symmetries (2.4) – (2.6) can be proven in the following theorem:

Theorem 1. If $u = u(x,y,t)$ is a solution of the ANNV equation (2.2), then

$$u' = \tau_t^{\frac{1}{3}} u(\xi, \eta, \tau) + \int \tau_t^{\frac{1}{3}} h(t)\mathrm{d}t + \frac{1}{18}\frac{\tau_{tt}\left(\int k(t)\mathrm{d}t\right)^2}{\tau_t} + \frac{1}{3}k(t)\int k(t)\mathrm{d}t - \frac{1}{3}\int k(t)^2\mathrm{d}t + \frac{1}{9}\frac{x\tau_{tt}\int k(t)\mathrm{d}t}{\tau_t} + \frac{1}{3}xk(t) + \frac{1}{18}\frac{x^2\tau_{tt}}{\tau_t}, \tag{2.13}$$

where

$$\xi = \left(\int k(t)\mathrm{d}t + x\right)\tau_t^{1/3}, \ \eta = \eta(y), \ \tau = \tau(t), \tag{2.14}$$

and $\{\tau = \tau(t), h = h(t), k = k(t), \eta = \eta(y)\}$ are all arbitrary functions of the indicated variable respectively, is also a solution of the ANNV equation (2.2).

We omit the detailed proof of the theorem because it can be directly verified by substituting (2.13) into (2.2).

3. An ANNV Family with a Common Kac-Moody-Virasoro Symmetry Algebra

In this section, we look for the possible equations which possess the same symmetries (2.4) – (2.6) and the same Kac-Moody-Virasoro symmetry algebra (2.8) – (2.11).

3.1. Models with the Symmetry (2.6)

The symmetry equation of (1.2) has the form

$$\sigma_{yt} + \sigma_{xxxy} - 3\sigma_{xx}u_y - 3u_{xx}\sigma_y - 3\sigma_x u_{xy} - 3u_x\sigma_{xy} + F'\sigma = 0, \tag{3.1}$$

where F' is the linearized operator of F defined by

$$F'G = \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}F(u+\varepsilon G) \tag{3.2}$$

for an arbitrary function G. Substituting (2.6) into (3.1) and eliminating u_{yt} from (1.2) yields

$$F'(x\dot{f}u_x) - x\dot{f}F'u_x + 3F'(fu_t) - 3fF'u_t - 4\dot{f}F + F'(u\dot{f} + \frac{1}{6}x^2\ddot{f}) = 0. \tag{3.3}$$

According to the definition of the function F (1.3) and its linearized operator F' (3.2), we have

$$F' = \sum_{n,m}\frac{\partial F}{\partial u_{x^n y^m}}\frac{\partial^{n+m}}{\partial x^n\partial y^m}, \qquad n,m \geq 0. \tag{3.4}$$

The substitution (3.4) into (3.3) gives

$$\sum_{n,m}\dot{f}\frac{(n+1)\partial F}{\partial u_{x^n y^m}}u_{x^n y^m} + \frac{1}{6}\ddot{f}\left[2\frac{\partial F}{\partial u_{xx}} + 2x\frac{\partial F}{\partial u_x} + x^2\frac{\partial F}{\partial u}\right] = 4\dot{f}F. \tag{3.5}$$

Under the f-independent requirement and the autonomous condition of F, (3.5) is equivalent to the equations

$$\frac{\partial F}{\partial u} = 0, \quad \frac{\partial F}{\partial u_x} = 0, \quad \frac{\partial F}{\partial u_{xx}} = 0, \tag{3.6}$$

$$\sum_{n,m}(n+1)\frac{\partial F}{\partial u_{x^n y^m}}u_{x^n y^m}=4F. \tag{3.7}$$

The general solution of (3.6)-(3.7) reads

$$\begin{aligned} F &= u_{xy}^2 F_1(v_{nm}, n, m = 0, 1, 2, \cdots) \\ &\equiv u_{xy}^2 F_1, \end{aligned} \tag{3.8}$$

$$\begin{aligned} v_{nm} &= v_{n,m} \\ &\equiv (1-\delta_{n0}\delta_{0m}-\delta_{n1}\delta_{0m}-\delta_{n2}\delta_{0m}) \\ &\quad \cdot u_{x^n y^m} u_{xy}^{-(n+1)/2}, \end{aligned} \tag{3.9}$$

where

$$\delta_{nm}=\begin{cases} 0 & n\neq m \\ 1 & n=m \end{cases}.$$

In summary, it has been proven that the equation

$$\begin{aligned} &u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_x u_{xy} \\ &\quad +u_{xy}^2 F_1(v_{nm}, n, m=0,1,2,\cdots)=0 \end{aligned} \tag{3.10}$$

with an arbitrary function of v_{nm} possesses a common Virasoro symmetry algebra (2.6).

3.2. Models with the Symmetry (2.5)

Substituting (2.5) into (3.1) gives

$$\begin{aligned} &g u_{xyt}+g u_{xxxxy}-6g u_{xx}u_{xy}-3g u_{xxx}u_y \\ &\quad -3g u_x u_{xxy}+F'(g u_x+\frac{x}{3}\dot{g})=0. \end{aligned} \tag{3.11}$$

Then eliminating u_{yt} from (1.2), we obtain

$$F'(g u_x+\frac{1}{3}x\dot{g})-gF'u_x=0. \tag{3.12}$$

From (3.4) and (3.12) follows

$$\frac{\partial F}{\partial u_x}+x\frac{\partial F}{\partial u}=0. \tag{3.13}$$

The general autonomous solution of (3.13) reads

$$\begin{aligned} F &= F_2 \equiv F_2((1-\delta_{n0}\delta_{0m}-\delta_{n1}\delta_{0m})u_{x^n y^m}, \\ &\quad \{n,m\}=0,\ 1,\ 2,\ \ldots). \end{aligned} \tag{3.14}$$

Obviously, $F=u_{xy}^2F_1(v_{nm},n,m=0,\ 1\ ,\ 2,\ \cdots)$ shown by (3.8) is only a special case of F_2 of (3.14).

3.3. Models with the Symmetry (2.4)

Substituting (2.4) into (3.1) and eliminating u_{yt} from (1.2) yields

$$F'h=0. \tag{3.15}$$

Substituting (3.4) into (3.15) gives

$$\frac{\partial F}{\partial u}=0. \tag{3.16}$$

It is clear that the general solution of (3.16) has the form

$$\begin{aligned} F &= F_3(u_{x^n y^m}, n, m = 0, 1, 2, \cdots, \\ &\quad \{n,m\}\neq\{0,0\}), \end{aligned} \tag{3.17}$$

and (3.8) is a special case of (3.14) and/or (3.17). In other words, the general ANNV family

$$\begin{aligned} &u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_x u_{xy} \\ &\quad +u_{xy}^2F_1(v_{nm},n,m=0,\ 1,\ 2,\cdots)=0 \end{aligned} \tag{3.18}$$

possesses not only the Virasoro symmetry algebra (2.8) but also the Kac-Moody-Virasoro symmetry algebra constructed by $\sigma_1(h),\ \sigma_2(g),\ \sigma_3(f)$.

4. Group Invariant Solutions of the ANNV Family

To find group invariant solutions of a given system means to find the solutions which are solutions of not only the original model but also the symmetry constrained condition $\sigma=0$. Because (3.18) possesses a common Kac-Moody-Virasoro symmetry algebra constituted by $\sigma_1-\sigma_3$, in this section we only consider the possible similarity reductions related to the symmetries $\sigma_1-\sigma_3$. If the field u satisfies either the model equation or the symmetry constrained condition

$$\sigma_1(h)+\sigma_2(g)+\sigma_3(f)=0 \tag{4.1}$$

with $\{\sigma_1(h),\sigma_2(g),\sigma_3(f)\}$ being given by (2.4), (2.5), (2.6), then the solution is invariant under the Kac-Moody-Virasoro group transformations. The symmetry constrained equation (4.1) can be solved easily because it is a linear equation and one can obtain the solutions by solving the characteristic equation.

We consider two cases

Case (i): $f \neq 0$.

For the full Kac-Moody-Virasoro symmetry algebra, $f \neq 0$, the general solution of (4.1) reads

$$u = \frac{U(\xi_1,\xi_2)}{9f^{1/3}} - \frac{xg}{9f} + \frac{1}{27f^{1/3}}\int \frac{g^2}{f^{5/3}}\mathrm{d}t - \frac{\dot{f}x^2}{18f} + \frac{1}{3}\frac{1}{f^{1/3}}\int \frac{h}{f^{1/3}}\mathrm{d}t, \tag{4.2}$$

where

$$\xi_1 = y, \quad \xi_2 = \frac{3x}{f^{1/3}} - \int \frac{g}{f^{4/3}}\mathrm{d}t,$$

while the invariant solution, $U(\xi_1,\xi_2) \equiv U$, should be determined by

$$27U_{\xi_1\xi_2\xi_2\xi_2} - 3U_{\xi_2}U_{\xi_1\xi_2} - 3U_{\xi_1}U_{\xi_2\xi_2} + U^2_{\xi_1\xi_2}F_1\left(3^{(3n-3)/2}U_{\xi_2^n\xi_1^m}U_{\xi_1\xi_2}^{-(n+1)/2}\right) = 0.$$

Case (ii): $f = 0$.

For the Kac-Moody symmetries, $f = 0$, the general solution of (4.1) has the form

$$\xi_1 = t, \quad \xi_2 = y, \quad u = \frac{U(\xi_1,\xi_2)}{g} - \frac{x^2}{6}\frac{\dot{g}}{g} + \frac{xh}{g}.$$

The similarity solution, $U(\xi_1,\xi_2) \equiv U$, should satisfy a wave equation

$$U_{\xi_1\xi_2} = 0.$$

5. Summary and Discussion

In summary, starting from the Kac-Moody-Virasoro symmetry of the usual ANNV equation and using a new symmetry approach proposed for the KP equation [5], a general ANNV family (3.18) is found such that the family possesses the same Kac-Moody-Virasoro symmetry algebra. An arbitrary function of the higher order group invariants is included in the family. Using the common symmetry algebra, the similarity reductions of the general ANNV system are obtained via the standard group approach.

From the general ANNV family (3.18), we know that in addition to the ANNV equation itself, one may obtain various models in the family that are rational in the derivatives, say, by selecting F_1 of (3.18) being a rational function function of $v_{2k+1,m}$, $v_{2k,m}v_{2l,p}$, $k,l = 1,2,\ldots$, $m = 0,1,2,\ldots$ etc. Though we have found many arbitrary order equations which possess the same Lie point symmetry groups of the KP equation [5] and the ANNV equation, we have not yet found any new integrable systems. How to find possible significant models (significant both in integrable theory and in real physical applications) from the equation families which possess common generalized Virasoro symmetry algebra should be studied in future.

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